ISSN No: -2456-2165 https://doi.org/10.38124/ijisrt/25sep1050

Three-Step Exponentially Fitted Two Hybrid Optimized Points for the Solution of Fourth Order Ordinary Differential Equations (ODEs)

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Publication Date: 2025/09/27

Abstract: This article presents an exponential fitted optimized three-step, two-off-grid hybrid point for the solution of fourth order ordinary differential equations. The method uses an exponential function as the basis function for a chosen two hybrid points, appropriately optimizing one of the two off-grid points by setting the principal term of the local truncation error to zero and using the local truncation error to determine the approximate values of the unknown parameter. Basic properties were examined, and the developed method was experimented to work out some fourth order initial value problems of ordinary differential equations. From the numerical results, it is clear that our new approach provides a better approximation than the existing method when compared to our result.

Keywords: *Three-Step, Optimization, Free-Parameter, Exponential Function, Fourth Order.*

How to Cite: Nuhu Bata Malgwi; Skwame Yusuf; Donald John Zirra (2025) Three-Step Exponentially Fitted Two Hybrid Optimized Points for the Solution of Fourth Order Ordinary Differential Equations (ODEs). *International Journal of Innovative Science and Research Technology,* 10 (9), 1798-1805. https://doi.org/10.38124/ijisrt/25sep1050

I. INTRODUCTION

In sciences and engineering, mathematical models are developed to understand as well as to interpret physical phenomena, many of such phenomena, when modeled, often result into higher order ordinary differential equations of the form:

$$\varphi^{iv} = g(x,\varphi,\varphi',\varphi'',\varphi'''),\quad \varphi(a_0)=\varphi(0),\varphi'(a_0)=\varphi'(0),\varphi''(a_0)=\varphi''(0),\varphi'''(a_0)=\varphi'''(0) \tag{1}$$

An essential mathematical tool for modeling a range of physical processes that occur in several scientific and engineering fields is the fourth order ordinary differential equation, including fluid dynamics, vibration analysis, control systems, and structural mechanics. According to [1], the fourth derivative can be optimized to predict, improve, and enhance computational efficiency and solution accuracy with superior accuracy and a good region of absolute stability. Equation (1) is used to solve problems in a number of real-world domains that frequently occur in physical systems in science and engineering, such as mechanics, control theory, beam deflection, and ship dynamics, among others. However, the majority of these physical issues are complex systems for which an analytical solution is exceedingly challenging, if not possible. Numerical techniques, which solve (1), are therefore essential tools. As a result, fourth order ordinary differential equations have garnered a lot of attention from researchers, and as a result, theoretical and numerical studies addressing (1) have recently surfaced in literature. As demonstrated by [2] and [3]. An old conventional way to solve (1) is the method of first reducing (1) to system of first order differential equation, to solve the resulting system of equations by any of the existing methods of solving first order ordinary differential equations. Literatures abounded in this old conventional method of solving problems of type (1) numerically are [4]-[6]. The drawbacks of this method include computational cumbersomeness and longer computer tine and space. In addition, [7] observes that these methods do not utilize additional information associated with a specific ordinary differential equation, such as oscillatory nature of the solution. To circumvent these drawbacks, many researchers have solved (1) directly; amongst these are [8]-[10] who develop blocked methods for numerical solution of fourth order ordinary differential equations. Among those who have recently embraced the hybrid approach as an alternative to the direct method for approximating (1) are [11], [12], and [13]. [1] have also proposed an optimized approach by using a novel fourth-order block algorithm to numerically solve fourth-order initial value problems; [14] have proposed an optimization of a two-step block method with three hybrid points for solving third-order initial value problems. [15] have proposed an optimization of a one-step

ISSN No: -2456-2165 https://doi.org/10.38124/ijisrt/25sep1050

block method with three hybrid points to solve first-order ordinary differential equations; [16] have also proposed an adaptive optimized step size hybrid method for integrating differential systems; and [17] has used the optimized approach to derive a two-step second derivative method for solving stiff systems. The three-step method created for this article, optimizes one of the two off-point placements by setting the principal term of the local truncation error to zero and using the local truncation error to estimate the values of the unknown parameter.

II. DERIVATION OF THE METHOD

We take our basis function to be exponentially fitted of the form

$$\varphi(t) = \sum_{j=0}^{(r+s)-1} \psi_j(t) e^{x^j} \tag{2}$$

Where $\psi \in \Re$ are real unknown coefficients that will be determined, while $r+s$ denotes number of collation and interpolation points.

The first, second, third and fourth derivatives of (2) are given by

$$\begin{aligned}
&\varphi'(t) = \sum_{j=1}^{s} j x^{j-1} \psi_j(t) e^{x^j} = g(t,\varphi)\\
&\varphi''(t) = \sum_{j=2}^{s} j x^{j-2} \psi_j(t) e^{x^j}\left(j + jx^j - 1\right) = g(t,\varphi,\varphi')\\
&\varphi'''(t) = \sum_{j=3}^{s} j x^{j-3} \psi_j(t) e^{x^j}\left(j^2 x^{2j} - 3j - 3jx^j + 3j^2 x^j + j^2 + 2\right) = g(t,\varphi,\varphi',\varphi'')\\
&\varphi^{iv}(t) = \sum_{j=4}^{s} j x^{j-4} \psi_j(t) e^{x^j}\left(11j - 6j^2 x^{2j} + 6j^3 x^{2j} + j^3 x^{3j} + 11jx^j - 18j^2 x^j + 7j^3 x^j - 6j^2 + j^3 + 6\right) = g(t,\varphi,\varphi',\varphi'',\varphi''')
\end{aligned} \tag{3}$$

Substituting (3) into (1) gives

$$^{j}\varphi''''(t) = f\left(t, {}^{j}\varphi, {}^{j}\varphi', {}^{j}\varphi'', {}^{j}\varphi'''\right) = \tau_0\varphi + h\tau_1\varphi' + h^2\tau_2\varphi'' + h^3\tau_3\varphi''' + \sum_{i=4}^{8} i(i-1)(i-2)(i-3)\psi_i t^{i-4}, \; j = 1,\dots,4 \tag{4}$$

Now interpolating (2) at first, second and third derivative at t_n and collocating (4) at all points $t_{n+\psi} = t_n + \psi h$, $\psi = \{0, r, s, 1, 2, 3\}$, give a nonlinear equation of the form

$$\begin{bmatrix}
6 & 6t_n & 3t_n^2 & t_n^3 & \frac{1}{4}t_n^4 & \frac{1}{20}t_n^5 & \frac{1}{120}t_n^6 & \frac{1}{840}t_n^7 & \frac{1}{6720}t_n^8 & \frac{1}{60480}t_n^9\\
0 & 6 & 6t_n & 3t_n^2 & t_n^3 & \frac{1}{4}t_n^4 & \frac{1}{20}t_n^5 & \frac{1}{120}t_n^6 & \frac{1}{840}t_n^7 & \frac{1}{6720}t_n^8\\
0 & 0 & 6 & 6t_n & 3t_n^2 & t_n^3 & \frac{1}{4}t_n^4 & \frac{1}{20}t_n^5 & \frac{1}{120}t_n^6 & \frac{1}{840}t_n^7\\
0 & 0 & 0 & 6 & 6t_n & 3t_n^2 & t_n^3 & \frac{1}{4}t_n^4 & \frac{1}{20}t_n^5 & \frac{1}{120}t_n^6\\
0 & 0 & 0 & 0 & 6 & 6t_n & 3t_n^2 & t_n^3 & \frac{1}{4}t_n^4 & \frac{1}{20}t_n^5\\
0 & 0 & 0 & 0 & 6 & 6t_n & 3t_{n+r}^2 & t_{n+r}^3 & \frac{1}{4}t_{n+r}^4 & \frac{1}{20}t_{n+r}^5\\
0 & 0 & 0 & 0 & 6 & 6t_n & 3t_{n+s}^2 & t_{n+s}^3 & \frac{1}{4}t_{n+s}^4 & \frac{1}{20}t_{n+s}^5\\
0 & 0 & 0 & 0 & 6 & 6t_n & 3t_{n+1}^2 & t_{n+1}^3 & \frac{1}{4}t_{n+1}^4 & \frac{1}{20}t_{n+1}^5\\
0 & 0 & 0 & 0 & 6 & 6t_n & 3t_{n+2}^2 & t_{n+2}^3 & \frac{1}{4}t_{n+2}^4 & \frac{1}{20}t_{n+2}^5\\
0 & 0 & 0 & 0 & 6 & 6t_n & 3t_{n+3}^2 & t_{n+3}^3 & \frac{1}{4}t_{n+3}^4 & \frac{1}{20}t_{n+3}^5
\end{bmatrix}^{-1}
\begin{bmatrix}
6\\
6\frac{(m)^1}{1!}\\
6\frac{(m)^2}{2!}\\
6\frac{(m)^3}{3!}\\
6\frac{(m)^4}{4!}\\
6\frac{(m)^5}{5!}\\
6\frac{(m)^6}{6!}\\
6\frac{(m)^7}{7!}\\
6\frac{(m)^8}{8!}
\end{bmatrix}
=
\begin{bmatrix}
\varphi_n\\
\varphi'_n\\
\varphi''_n\\
\varphi'''_n\\
g_n\\
g_{n+r}\\
g_{n+s}\\
g_{n+1}\\
g_{n+2}\\
g_{n+3}
\end{bmatrix} \tag{5}$$

Applying the Guassian elimination method to solve equation (5) gives the coefficient $\tau_i,(j=0)$ *and* $\psi_j,(0,r,s,1,2,3)$. The values are then substituted into equation (2) to give the implicit continuous hybrid method of the form;

$$\varphi(x)=\tau_0\varphi_n+\tau_1 h\varphi'_n+\tau_2 h^2\varphi''_n+\tau_3 h^3\varphi'''_n+h^4\left[\sum_{j=0}^{3}\psi_j g_{n+j}+\psi_{\varsigma j}g_{n+\varsigma j}\right],\varsigma=r,s \tag{6}$$

were

$$\tau_0=-\frac{1}{90720}m^4\left(\frac{-462rm^2+108rm^3-462m^2s-9m^4s+108sm^3-9m^4r+756ms-3780rs-252m^2+198m^3-54m^4+5m^5-252m^2rs+18m^3rs+1386mrs}{rs}\right)$$

$$\tau_1=\frac{1}{15120}m^5\left(\frac{-252m+756s+108m^2s-9m^3s-462ms+198m^2-54m^3+5m^4}{r(r-3)(r-1)(r-2)(r-s)}\right)$$

$$\tau_2=-\frac{1}{15120}m^5\left(\frac{-252m+756s+108m^2s-9m^3s-462ms+198m^2-54m^3+5m^4}{s(s-3)(s-1)(s-2)(r-s)}\right)$$

$$\tau_3=\frac{1}{30240}m^5\left(\frac{90m^2r-9m^3r+90m^2s-9m^3s-252mr-252ms+756rs+108m^2-45m^3+5m^4+18m^2rs-210mrs}{(s-1)(r-1)}\right)$$

$$\tau_4=\frac{1}{30240}m^5\left(\frac{72m^2r-9m^3r+72m^2s-9m^3s-126mr-126ms+378rs+54m^2-36m^3+5m^4+18m^2rs-168mrs}{(s-2)(r-2)}\right)$$

$$\tau_5=\frac{1}{90720}m^5\left(\frac{54m^2r-9m^3r+54m^2s-9m^3s-84mr-84ms+252rs+36m^2-27m^3+54m^4+18m^2rs-126msr}{(s-3)(r-3)}\right)$$

by substituting $m=1$ in (6) we obtain a multistep formula to approximate the solution of (1) at the point t_{n+1} which yields

$$t_{n+1}=\varphi_n+h\varphi'_n+\tfrac{1}{2}\varphi''_n h^2+\tfrac{1}{6}h^3\varphi'''_n+h^4\begin{bmatrix}\left(\frac{1}{90720}\frac{(-393r-393s+2628rs+103)}{rs}\right)g_n+\left(\frac{1}{15120}\frac{(393s-103)}{r(r-3)(r-1)(r-2)(r-s)}\right)g_{n+r}\\-\left(\frac{1}{15120}\frac{(393s-103)}{s(s-3)(s-1)(s-2)(r-s)}\right)g_{n+s}+\left(\frac{1}{30240}\frac{(-171r-171s+564rs+68)}{(s-1)(r-1)}\right)g_{n+1}\\-\left(\frac{1}{30240}\frac{(-63r-63s+228rs+23)}{(s-2)(r-2)}\right)g_{n+2}+\left(\frac{1}{90720}\frac{(-39r-39s+144rs+14)}{(s-3)(r-3)}\right)g_{n+3}\end{bmatrix} \tag{7}$$

Also by substituting $m=1$ in the first derivative of (6) we obtain a multistep formula to approximate the solution of (1) at the point t'_{n+1} which yields

$$ht'_{n+1}=h\varphi'_n+\varphi''_n h^2+\tfrac{1}{2}h^3\varphi'''_n+h^4\begin{bmatrix}\left(\frac{1}{10080}\frac{(-188r-188s+1064rs+57)}{rs}\right)g_n+\left(\frac{1}{1680}\frac{(188s-57)}{r(r-3)(r-1)(r-2)(r-s)}\right)g_{n+r}\\-\left(\frac{1}{1680}\frac{(188r-57)}{s(s-3)(s-1)(s-2)(r-s)}\right)g_{n+s}+\left(\frac{1}{3360}\frac{(-106r-106s+294rs+49)}{(s-1)(r-1)}\right)g_{n+1}\\-\left(\frac{1}{3360}\frac{(-36r-36s+112rs+15)}{(s-2)(r-2)}\right)g_{n+2}+\left(\frac{1}{10080}\frac{(-22r-22s+70rs+9)}{(s-3)(r-3)}\right)g_{n+3}\end{bmatrix} \tag{8}$$

Also by substituting $m=1$ in the second derivative of (6) we obtain a multistep formula to approximate the solution of (1) at the point t''_{n+1} which yields

$$h^2t''_{n+1}=\varphi''_n h^2+h^3\varphi'''_n+h^4\begin{bmatrix}\left(\frac{1}{2520}\frac{(-147r-147s+679rs+53)}{rs}\right)g_n+\left(\frac{1}{420}\frac{(147s-53)}{r(r-3)(r-1)(r-2)(r-s)}\right)g_{n+r}\\-\left(\frac{1}{420}\frac{(147r-53)}{s(s-3)(s-1)(s-2)(r-s)}\right)g_{n+s}+\left(\frac{1}{840}\frac{(-119r-119s+266rs+66)}{(s-1)(r-1)}\right)g_{n+1}\\-\left(\frac{1}{840}\frac{(-35r-35s+91rs+17)}{(s-2)(r-2)}\right)g_{n+2}+\left(\frac{1}{2520}\frac{(-21r-21s+56rs+10)}{(s-3)(r-3)}\right)g_{n+3}\end{bmatrix} \tag{9}$$

ISSN No: -2456-2165 https://doi.org/10.38124/ijisrt/25sep1050

Finally, by substituting $m=1$ in the third derivative of (6) we obtain a multistep formula to approximate the solution of (1) at the point t'''_{n+1} which yields

$$h^3 t'''_{n+1} = \varphi'''_n h^3 + h^4 \begin{bmatrix} \left(\frac{1}{360}\frac{(-38r-38s+135rs+17)}{rs}\right) g_n + \left(\frac{1}{60}\frac{(38s-17)}{r(r-3)(r-1)(r-2)(r-s)}\right) g_{n+r} \\ -\left(\frac{1}{60}\frac{(38s-17)}{s(s-3)(s-1)(s-2)(r-s)}\right) g_{n+s} + \left(\frac{1}{120}\frac{(-57r-57s+95rs+40)}{(s-1)(r-1)}\right) g_{n+1} \\ -\left(\frac{1}{120}\frac{(-12r-12s+25rs+7)}{(s-2)(r-2)}\right) g_{n+2} + \left(\frac{1}{360}\frac{(-7r-7s+15rs+4)}{(s-3)(r-3)}\right) g_{n+3} \end{bmatrix} \tag{10}$$

➢ *Derivation of the Optimize Method*

The derivation for the implementation of the optimized three step at third derivative is given by

- Expanding (10) using the Taylor series to obtain the corresponding Local Truncation Error:

$$L\left(\varphi\left(t_{j+1}\right), h\right) = -\frac{\left(-119r-119s+266rs+66\right)h^8}{50400} + oh^9 \tag{11}$$

- Equating the principal term of the Local Truncation Error in (11) to zero, and keeping s as a free parameter and assigning the value $s = \dfrac{2}{3}$, we obtain the value $r = \frac{8}{35}$ as the optimized value.

- Substituting the values of r and s into (7) – (10) produces the following general equations in block form

$$C^{[0]}\varphi_m^{[1]} = C^{[1]}\varphi_m^{[0]} + \sum_{i=0}^{k} D^{[i]}\Gamma_m^{[i]}, i = \{0, r, s, 1, 2, 3\} \tag{12}$$

this gives a discrete schemes:

$$\varphi_{n+\frac{8}{35}} = \varphi_n + \frac{8}{35} h\varphi'_n + \frac{32}{1225} h^2\varphi''_n + \frac{256}{128625} h^3\varphi'''_n + \frac{56922990592}{709340748046875} h^4 g_n + \frac{2493219328}{58845939429375} h^4 g_{n+\frac{8}{35}}$$
$$- \frac{179400801792}{126893178261718 75} h^4 g_{n+\frac{2}{3}} + \frac{33649983488}{6384066732421875} h^4 g_{n+1} - \frac{6105325568}{219895631894531 25} h^4 g_{n+2}$$
$$+ \frac{3579969536}{1605474559746093 75} h^4 g_{n+3}$$

$$\varphi_{n+\frac{2}{3}} = \varphi_n + \frac{2}{3} h\varphi'_n + \frac{2}{9} h^2\varphi''_n + \frac{4}{81} h^3\varphi'''_n + \frac{58568}{18600435} h^4 g_n + \frac{1776740000}{330794919009} h^4 g_{n+\frac{8}{35}} - \frac{1258}{2464749} h^4 g_{n+\frac{2}{3}} + \frac{39016}{167403915} h^4 g_{n+1}$$
$$- \frac{2528}{192204495} h^4 g_{n+2} + \frac{13528}{1262965365} h^4 g_{n+3}$$

$$\varphi_{n+1} = \varphi_n + h\varphi'_n + \frac{1}{2} h^2\varphi''_n + \frac{1}{6} h^3\varphi'''_n + \frac{1769}{161280} h^4 g_n + \frac{79533125}{2868244992} h^4 g_{n+\frac{8}{35}} + \frac{4149}{1442560} h^4 g_{n+\frac{2}{3}} + \frac{1}{9072} h^4 g_{n+1} - \frac{47}{2499840} h^4 g_{n+2}$$
$$+ \frac{1}{570360} h^4 g_{n+3}$$

$$\varphi_{n+2} = \varphi_n + 2h\varphi'_n + 2h^2\varphi''_n + \frac{4}{3} h^3\varphi'''_n + \frac{76}{945} h^4 g_n + \frac{6002500}{16806123} h^4 g_{n+\frac{8}{35}} + \frac{792}{5635} h^4 g_{n+\frac{2}{3}} + \frac{104}{1215} h^4 g_{n+1}$$
$$+ \frac{2}{651} h^4 g_{n+2} - \frac{88}{641655} h^4 g_{n+3}$$

ISSN No: -2456-2165 https://doi.org/10.38124/ijisrt/25sep1050

$$\varphi_{n+3} = \varphi_n + 3h\varphi'_n + \frac{9}{2}h^2\varphi''_n + \frac{9}{2}h^3\varphi'''_n + \frac{3051}{17920}h^4 g_n + \frac{58524375}{35410432}h^4 g_{n+\frac{8}{35}} + \frac{334611}{1442560}h^4 g_{n+\frac{2}{3}} + \frac{321}{280}h^4 g_{n+1} + \frac{48519}{277760}h^4 g_{n+2} - \frac{81}{76048}h^4 g_{n+3}$$

$$\varphi'_{n+\frac{8}{35}} = \varphi'_n + \frac{8}{35}h\varphi''_n + \frac{32}{1225}h^2\varphi'''_n + \frac{2891211968}{2251875390625}h^3 g_n + \frac{1476721984}{1681312555125}h^3 g_{n+\frac{8}{35}} - \frac{97193820672}{18432062528}h^3 g_{n+\frac{2}{3}} + \frac{18423062528}{182401906640625}h^3 g_{n+1} - \frac{1107923456}{209424411328125}h^3 g_{n+2} + \frac{1945542656}{4587070170703125}h^3 g_{n+3}$$

$$\varphi'_{n+\frac{2}{3}} = \varphi'_n + \frac{2}{3}h\varphi''_n + \frac{2}{9}h^2\varphi'''_n + \frac{6857}{459270}h^3 g_n + \frac{277615625}{8167775778}h^3 g_{n+\frac{8}{35}} - \frac{19}{182574}h^3 g_{n+\frac{2}{3}} + \frac{1256}{2066715}h^3 g_{n+1} - \frac{209}{4745790}h^3 g_{n+2} + \frac{584}{155922165}h^3 g_{n+3}$$

$$\varphi'_{n+1} = \varphi'_n + h\varphi''_n + \frac{1}{2}h^2\varphi'''_n + \frac{593}{17920}h^3 g_n + \frac{307628125}{2868244992}h^3 g_{n+\frac{8}{35}} + \frac{39771}{1442560}h^3 g_{n+\frac{2}{3}} - \frac{23}{18144}h^3 g_{n+1} + \frac{17}{833280}h^3 g_{n+2} - \frac{1}{2281440}h^3 g_{n+3}$$

$$\varphi'_{n+2} = \varphi'_n + 2h\varphi''_n + 2h^2\varphi'''_n + \frac{61}{630}h^3 g_n + \frac{7503125}{11204082}h^3 g_{n+\frac{8}{35}} + \frac{2349}{11270}h^3 g_{n+\frac{2}{3}} + \frac{968}{2835}h^3 g_{n+1} + \frac{23}{1302}h^3 g_{n+2} - \frac{152}{213885}h^3 g_{n+3}$$

$$\varphi'_{n+3} = \varphi'_n + 3h\varphi''_n + \frac{9}{2}h^2\varphi'''_n + \frac{2241}{17920}h^3 g_n + \frac{67528125}{35410432}h^3 g_{n+\frac{8}{35}} + \frac{177147}{1442560}h^3 g_{n+\frac{2}{3}} + \frac{2061}{1120}h^3 g_{n+1} + \frac{138267}{277760}h^3 g_{n+2} + \frac{1089}{152096}h^3 g_{n+3}$$

$$\varphi''_{n+\frac{8}{35}} = \varphi''_n + \frac{8}{35}h\varphi'''_n + \frac{8351129408}{579053671875}h^2 g_n + \frac{8390304}{593055575}h^2 g_{n+\frac{8}{35}} - \frac{39328841088}{10358626796875}h^2 g_{n+\frac{2}{3}} + \frac{273565696}{193017890625}h^2 g_{n+1} - \frac{440622464}{5983554609375}h^2 g_{n+2} + \frac{2315487232}{393177443203125}h^2 g_{n+3}$$

$$\varphi''_{n+\frac{2}{3}} = \varphi''_n + \frac{2}{3}h\varphi'''_n + \frac{5891}{131220}h^2 g_n + \frac{2528553125}{16335551556}h^2 g_{n+\frac{8}{35}} + \frac{16}{621}h^2 g_{n+\frac{2}{3}} - \frac{968}{295245}h^2 g_{n+1} + \frac{19}{338985}h^2 g_{n+2} - \frac{8}{3182085}h^2 g_{n+3}$$

$$\varphi''_{n+1} = \varphi''_n + h\varphi'''_n + \frac{373}{5760}h^2 g_n + \frac{7503125}{26557824}h^2 g_{n+\frac{8}{35}} + \frac{7857}{51520}h^2 g_{n+\frac{2}{3}} + \frac{7}{29760}h^2 g_{n+2} - \frac{1}{61110}h^2 g_{n+3}$$

$$\varphi''_{n+2} = \varphi''_n + 2h\varphi'''_n + \frac{1}{180}h^2 g_n + \frac{7503125}{7469388}h^2 g_{n+\frac{8}{35}} - \frac{81}{805}h^2 g_{n+\frac{2}{3}} + \frac{136}{135}h^2 g_{n+1} + \frac{8}{93}h^2 g_{n+2} - \frac{88}{30555}h^2 g_{n+3}$$

$$\varphi''_{n+3} = \varphi''_n + 3h\varphi'''_n + \frac{183}{640}h^2 g_n + \frac{7503125}{8852608}h^2 g_{n+\frac{8}{35}} + \frac{6561}{7360}h^2 g_{n+\frac{2}{3}} + \frac{13}{10}h^2 g_{n+1} + \frac{11043}{9920}h^2 g_{n+2} + \frac{6}{97}h^2 g_{n+3}$$

$$\varphi'''_{n+\frac{8}{35}} = \varphi'''_n + \frac{209807188}{2363484375}hg_n + \frac{31456708}{196071435}hg_{n+\frac{8}{35}} - \frac{1343171808}{42280109375}hg_{n+\frac{2}{3}} + \frac{248586752}{21271359375}hg_{n+1} - \frac{14644064}{24422671875}hg_{n+2} + \frac{76662272}{1604805890625}hg_{n+3}$$

ISSN No: -2456-2165 https://doi.org/10.38124/ijisrt/25sep1050

$$\varphi'''_{n+\frac{2}{3}} = \varphi'''_n + \frac{4771}{87480} hg_n + \frac{4359315625}{10890367704} hg_{n+\frac{8}{35}} + \frac{7141}{28980} hg_{n+\frac{2}{3}} - \frac{3512}{98415} hg_{n+1} + \frac{541}{451980} hg_{n+2} - \frac{632}{7424865} hg_{n+3}$$

$$\varphi'''_{n+1} = \varphi'''_n + \frac{373}{5760} hg_n + \frac{262609375}{717061248} hg_{n+\frac{8}{35}} + \frac{23571}{51520} hg_{n+\frac{2}{3}} + \frac{181}{1620} hg_{n+1} - \frac{7}{29760} hg_{n+2} + \frac{1}{122220} hg_{n+3}$$

$$\varphi'''_{n+2} = \varphi'''_n - \frac{79}{360} hg_n + \frac{52521875}{44816328} hg_{n+\frac{8}{35}} - \frac{3483}{3220} hg_{n+\frac{2}{3}} + \frac{728}{405} hg_{n+1} + \frac{631}{1860} hg_{n+2} - \frac{232}{30555} hg_{n+3}$$

$$\varphi'''_{n+3} = \varphi'''_n + \frac{621}{640} hg_n - \frac{52521875}{26557824} hg_{n+\frac{8}{35}} + \frac{194643}{51520} hg_{n+\frac{2}{3}} - \frac{101}{60} hg_{n+1} + \frac{16083}{9920} hg_{n+2} + \frac{3957}{13580} hg_{n+3} \quad (13)$$

III. ANALYSIS OF BASIC PROPERTIES OF THE METHOD

- *Order of the Block*

Let the linear difference operator $L\{\varphi(t): h\}$ associated with the new method (13) be express in the form

$$L\{\varphi(x):h\} = \sum_{j=0}^{N} p_j \varphi(t+jh) - h^4 \sum_{j=0}^{N} \left(g_j \varphi^{(4)}(t+jh) \right) \quad (14)$$

By expanding $\varphi(t + jh)$ and $g(t + jh)$ in Taylor series, (14) becomes:

$$L\{\varphi(x):h\} = C_0\varphi(x) + C_1\varphi'(x) + \cdots + C_p h^p \varphi^p(x) + C_{p+1} h^{p+1} \varphi^{p+1}(x) + \cdots + C_{p+4} h^{p+4} \varphi^{p+4}(x) + \cdots$$

According to Lambert, 1973, the linear operator L and its corresponding block formula are of order p, if $C_0 = C_1 = \cdots = C_p = C_{p+1} = C_{p+2} = C_{p+3} = 0$ and $C_{p+4} \neq 0$. The term C_{p+4} is called the error constant and implies that the local truncation error is given by:

$$t_{n+k} = C_{p+4} h^{p+4} y^{p+4}(x) + O(h^{p+5}).$$

Our method involves expanding (13) in the Taylor series, then comparing the coefficient of h yields

$$C_0 = C_1 = C_2 = C_3 = \cdots = C_9 = 0$$

Hence the block (13) has order 5 with error constant:

$$C_{10} = \begin{bmatrix} ?\frac{1242\,821\,607\,424}{351\,921\,678\,624\,755\,859\,375}, ?\frac{15\,217}{87\,887\,055\,375}, ?\frac{193}{571\,536\,000}, \frac{53}{4465\,125}, \frac{159}{784\,000} \\ ?\frac{4986\,438\,656}{74\,480\,778\,544\,921\,875}, ?\frac{1258}{1953\,045\,675}, ?\frac{257}{12\,700\,800}, ?\frac{467}{99\,225}, ?\frac{17\,131}{156\,800} \\ ?\frac{5906\,887\,936}{6384\,066\,732\,421\,875}, ?\frac{19}{21\,005\,075}, \frac{23}{12\,700\,800}, \frac{8863\,620\,180\,988\,492}{39\,318\,633\,167\,698\,828\,125} \\ ?\frac{144\,842\,602\,584\,717\,907}{186\,399\,446\,128\,350\,000\,000}, ?\frac{16\,780\,608}{2251\,875\,390\,625}, \frac{16}{1476\,225}, ?\frac{23}{88\,905\,600}, \frac{8}{14\,175}, ?\frac{8934\,287}{2622\,352\,320} \end{bmatrix}$$

- *Consistency of the Method*

According to (Lambert, 1991), the hybrid block method is said to be consistent if it has an order more than or equal to one. Therefore, our method is consistent.

- *Zero Stability of the Method*

The hybrid block (13) is said to be Zero stable if the roots $z_i, i = 0, r, s, 1, 2, 3$ of the first characteristic polynomial $\rho(z) = 0$ that is

ISSN No: -2456-2165 https://doi.org/10.38124/ijisrt/25sep1050

$\rho(z) = \det\left[\sum_{j=0}^{k} A^{(i)} z^{k-i}\right] = 0$, satisfies $|z_i| \leq 1$ and the roots $|z_i| = 1$, has multiplicity not exceeding the order of the differential equation. Hence, our method is zero-stable.

➢ *Convergence of the Method*

The necessary and sufficient condition for a numerical method to be convergent is for it to be consistent and Zero stable. Thus since it has been successfully shown from the above condition, it could be seen that our method is convergent.

➢ *Region of Absolute Stability of the Method*

The stability polynomial for the method is given as:

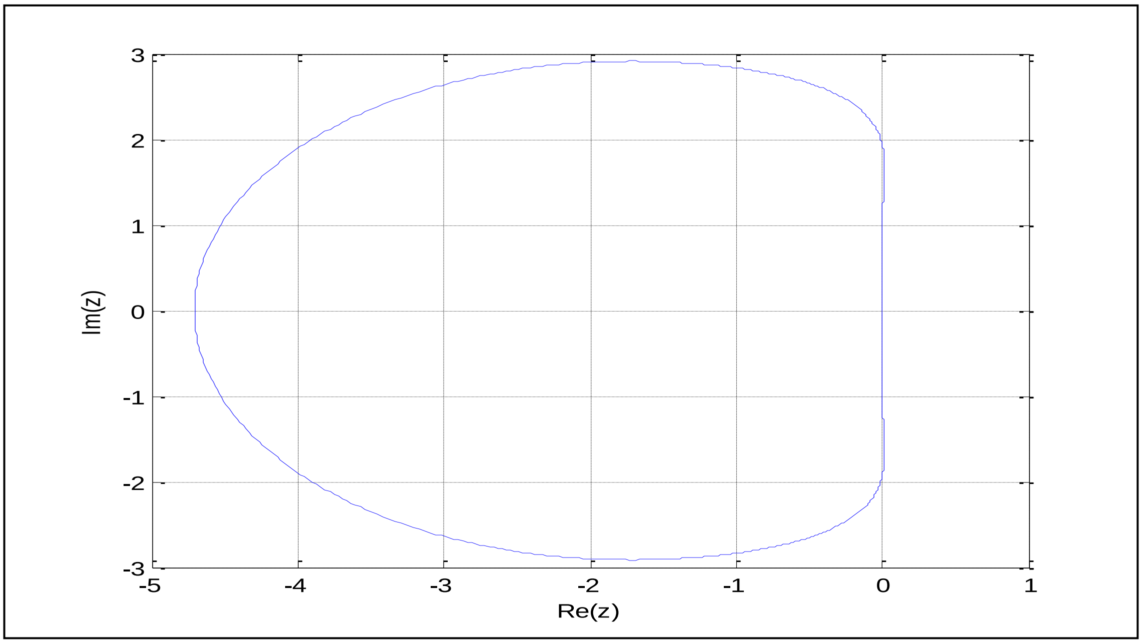

Fig 1 Region of Absolute Stability for the new Method

IV. NUMERICAL EXPERIMENTS

To validate the accuracy and suitability of the our method, we solve some initial value problems of fourth order ordinary differential equations and compare the results with the work.

Problem 1 $y^{iv} = x$

$$y(0) = 0, y'(0) = 1, y''(0) = 0, y'''(0) = 0,$$

Exact solution: $y(t) = \frac{x^5}{120} + x$

Table 1 Comparison of Absolute Error for Problem 1

t	An Error in Muritala *et al.*, (2023)	An Error in Akinnukawe *et al.*, (2024)	An Error in the New Method
0.1	0.0000e-00	1.3878e-17	0.0000
0.2	0.0000e-00	2.7756e-17	0.0000
0.3	0.0000e-00	5.5511e-17	1.0000e-20
0.4	0.0000e-00	5.5511e-17	2.0000e-20
0.5	0.0000e-00	0.0000	5.0000e-20
0.6	0.0000e-00	1.1102e-16	1.0000e-19
0.7	0.0000e-00	1.1102e-16	1.7000e-19
0.8	0.0000e-00	2.2205e-16	2.8000e-19
0.9	0.0000e-00	2.2205e-16	4.5000e-19
1.0	0.0000e-00	2.2205e-16	0.0000

Problem 2 $y^{iv} = 4y''$

ISSN No: -2456-2165 https://doi.org/10.38124/ijisrt/25sep1050

$$y(0) = 1, y'(0) = 3, y''(0) = 0, y'''(0) = 16$$

Exact solution: $y(t) = 1 - t + e^{2t} - e^{-2t}$

Table 2 Comparison of Absolute Error for Problem 1

t	**An Error in Awoyemi *et al.*, (2015)**	**An Error in Muritala et al., (2023)**	**An Error in the New Method**
0.003125	0.0000e+00	0.0000e+00	0.0000
0.006250	0.0000e+00	0.0000e+00	0.0000
0.009375	2.2205e-16	0.0000e+00	0.0000
0.012500	2.4425e-15	0.0000e+00	1.0000e-19
0.015625	1.1546e-14	1.0000e-20	0.0000
0.018750	3.3085e-14	0.0000e+00	0.0000
0.021875	7.2831e-14	1.0000e-20	1.0000e-19
0.025000	1.3700e-13	0.0000e+00	0.0000
0.028125	2.3093e-13	0.0000e+00	1.0000e-19
0.031250	3.6082e-13	1.0000e-20	0.0000

V. CONCLUSION

It is evident from the above tables that our proposed method; the optimized approaches can handle stiff equations and, in fact, converge faster than the compared method. Hence the approach has a significant improvement over the existing methods.

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