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A Comparative Analysis of Laplace Transform and Adomian Decomposition Methods for Solving the 3D Heat Equation: Accuracy, Efficiency, and Flexibility

Yasir Abeid Hussin[1,2]; Safa Khider Malik[1]; Abir Khalil Salibi[3]

[1]Department of Mathematics- College of Education - Omdurman Islamic University, Sudan
[2]Department of Mathematics -Faculty of Education - Gulf of Saidra University, Libya
[3]Janzour Department of Mathematics -Faculty of Education -University of Tripoli, Libya.

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Abstract: This paper presents a comparative analysis of the Laplace Transform Method and the Adomian Decomposition Method (ADM) for solving the three-dimensional heat equation. The Laplace Transform Method converts equations into the frequency domain, enabling precise solutions for linear systems but struggling with asymmetric and nonlinear cases. In contrast, ADM decomposes the solution into an infinite series computed recursively, making it suitable for complex nonlinear applications. Through comparative analysis, this paper demonstrates that the Laplace Transform Method offers high accuracy for linear cases, while ADM is more flexible and better suited for handling complex boundary conditions. Consequently, the Laplace Transform is preferable for simple linear problems, whereas ADM proves to be more effective for complex and nonlinear cases.

Keywords: *Adomian, Laplace, Transform, Decomposition.*

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I. INTRODUCTION

Modern mathematical techniques play a crucial role in solving partial differential equations, especially in physics and engineering. Among these techniques, Laplace Transform and Adomian Decomposition (ADM) are effective tools for solving the three-dimensional heat equation [1-3]. The Laplace Transform provides exact solutions in linear cases but struggles with asymmetric geometries [4,5], whereas ADM is highly adaptable to nonlinear systems without requiring complex transformations [1,2,6]. This paper aims to compare these two methods in terms of accuracy, efficiency, and their ability to handle geometric and physical complexities [6,7, and 8].

II. ADOMIAN DECOMPOSITION FORMULA

The Adomian decomposition method consists of decomposing the unknown function u(x,y) of an equation it sums of an infinite number of components defined by the decomposition series [1,2].

$$u(x\,,y) = \sum_{n=0}^{\infty} \mathrm{u_n}\ (\,x\,,y\,). \qquad (1)$$

where the component $u_n(x,y), n \geq 0$ are to be determined a recursive in manner the decomposition method concerns itself with find the components u_0, u_1, u_2 individually. As will be seen thorough recursive relations, these components are usually obtained through simple integrals [4,6].

To give a clear overview of Adomian Decomposition Method. we first consider the linear differential equation write in an operator form by

$$L(u) + R(u) = g \qquad (2)$$

Where L is an operator, typically representing a lower-order derivative which is assumed to be invertible, R is other linear differential equations will be represented in last chapter. We next apply the inverse operator $L{-}1$ to both sides of equation (1-2):

$$L^{-1}\left(L(u) + R(u)\right) = L^{-1}(g)$$

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$$\left(L^{-1}L(u)+L^{-1}R(u)\right)=L^{-1}(g)$$

$$\Longrightarrow u+L^{-1}R\,(u)=f \quad (3)$$

where f = L-1 (g), then we get:

$$u=f-L^{-1}R\,(u) \quad (4)$$

where function f represents the terms arising from integrating the source term g and from using the given condition that are assumed to be prescribed. As indicated before, the Adomian method defines the solution as an infinite series of components given by:

$$u=\sum_{n=0}^{\infty}u_n \quad (5)$$

where the components u_0, u_1, u_2 are usually recurrently determined substituting it to both

$$\sum_{n=0}^{\infty}u_n=f-L^{-1}\left(R\left(\sum_{n=0}^{\infty}u_n\right)\right) \quad (6)$$

To simplify, equation (6) can be written as follows:

$$u_0+u_1+u_2=f-L^{-1}\left(R(u_0+u_1+u_2\dots\right) \quad (7)$$

The inverse operator of time derivative, denoted as L^{-1}, is defined as the integral with respect to time:

$$L^{-1}\left(\frac{\partial u}{\partial t}\right)=\int_0^t u(t)\,.\,\mathrm{dt} \quad (8)$$

It is used to transform the differential equation, allowing the application of the Adomion Decomposition method to solve the heat equation analytically and approximately [2,5, and 6].

III. THREE-DIMENSIONAL HEAT FLOW

The distribution of heat flow in a three-dimensional space is governed by the following initial boundary value problem:

$$\left\{\begin{array}{ll} PDE & u_t=k\left(u_{xx}+u_{yy}+u_{zz}\right), t>0, \\ & 0<x<a, 0<y<b, 0<z<c \\ BC & u(0,y,z,t)=u(a,y,z,t)=0 \\ & u(x,0,z,t)=u(x,b,z,t)=0 \\ & u(x,y,0,t)=u(x,y,c,t) \\ IC & u(x,y,z,0)=F(x,y,z) \end{array}\right\} \quad (9)$$

Where u=u (x,y,z,t, t, t) is the temperature of any point located at the position (x, y, z) of a rectangular volume at any time t, and k¯is the thermal diffusivity we first rewrite in an operator form by:

$$L_t u=\bar{k}\left(L_x u+L_y u+L_z u\right) \quad (10)$$

Where the differential operators L_x, L_y and L_z are defined by:

$$L_t=\frac{\partial}{\partial t}, \quad L_x=\frac{\partial^2}{\partial x^2}, L_y=\frac{\partial^2}{\partial y^2}, L_z=\frac{\partial^2}{\partial z^2} \quad (11)$$

So that the integral operator L_t^{-1} exists and given by:

$$L_t^{-1}(.)=\int_0^t(.)dt \quad (12)$$

Applying L_t^{-1} to both sides of using the initial condition leads to:

$$u\,(x,y,z,t)=F\,(x,y,z)+\bar{k}\,L_t^{-1}(L_x u+L_y u+L_z u)$$

The decomposition method defines the solution u(x ,y ,z ,t) as a series given by:

$$u\,(x,y,z,t)=\sum_{n=0}^{\infty}u_n\,(x,y,z,t) \quad (13)$$

Substituting into both sides of

$$\sum_{n=0}^{\infty}u_n=F(x,y,z)+\bar{k}L_t^{-1}\left(L_x\left(\sum_{n=1}^{\infty}u_n\right)+L_y\left(\sum_{n=1}^{\infty}u_n\right)+L_z\left(\sum_{n=1}^{\infty}u_n\right)\right)$$

The components $u_n(x,y,z,t)$, $n\geq 0$ can be completely determined by using the recursive relationship:

$$\left.\begin{array}{c} u_0\,(x,y,z,t)=F(x,y,z) \\ u_{k+1}(x,y,z,t)=\bar{k}L_t^{-1}\left(L_x u_k+L_y u_x+L_z u_k\right), k\geq 0 \end{array}\right\}$$

The components can be determined recursively as do as we like.

Consequently, the components $u_n, n\geq 0$ are completely determined and the solution in a series form follows immediately [1,2, and 6].

IV. LAPLACE TRANSFORM

The Laplace transform of a function u(x,y,z,t) with respect to time t is defined as:

$$L\{u(x,y,z,t\}=u(x,y,z,s)$$

$$=\int_0^{\infty}u(x,y,z,t)e^{-st}.\,dt \quad (14)$$

Where s is a complex variable in the Laplace domain.

A.Linearity of the Laplace Transform:

The Laplace transform is linear, meaning: $L\{a\,u+bv\}=aL\{u\}+bL\{v\}$ For constants ɑ and b, and function u and v.

B. Transformation of time Derivatives:

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➢ *First – order time derivatives:*

$$L\left\{\frac{\partial u}{\partial t}\right\} = S\,U(x,y,z,s) - u(x,y,z,0)$$

where u(x,y,z,0) is the initial condition.

➢ *Second – order time derivatives:*

$$L\left\{\frac{\partial^2 u}{\partial t^2}\right\} = S^2U(x,y,z,s) - Su(x,y,z,0) - \frac{\partial u}{\partial t}(x,y,z,0)$$

C.Spatial Derivatives Remain Unchanged:

$$L\left\{\frac{\partial u}{\partial x}\right\} = \frac{\partial u}{\partial x}$$

$$L\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \frac{\partial^2 u}{\partial x^2}$$

$$L\left\{\frac{\partial u}{\partial y}\right\} = \frac{\partial u}{\partial y}$$

$$L\left\{\frac{\partial^2 u}{\partial y^2}\right\} = \frac{\partial^2 u}{\partial y^2}$$

$$L\left\{\frac{\partial u}{\partial z}\right\} = \frac{\partial u}{\partial z}$$

$$L\left\{\frac{\partial^2 u}{\partial z^2}\right\} = \frac{\partial^2 u}{\partial z^2}$$

V. COMPARISON BETWEEN LAPLACE TRANSFORM AND ADOMIAN DECOMPOSITION METHODS FOR SOLVING 3D HEAT EQUATION

➢ *Which is Better and why?*

- *Laplace Transform method:*

Methodology: Converts the time - dependent PDE into Frequency - domain ODE (Helmholtz equation)

$$\nabla^2 U - \frac{s}{\alpha}U = -\frac{F(x\,,y\,,z)}{\alpha}$$

Solves the ODE and applies the inverse Laplace transform 1,7, and 8].

➢ *Advantages:*

- High Accuracy: Provides exact analytical solutions if the inverse transforms is tractable. Efficiency for linear Cases: [deal for linear equations with constant coefficients [9].
- Inverse Transform complexity: Challenging in asymmetric 3D geometries.
- Inflexibility: Fails for nonlinear equations or variable coefficients.

- *Adomian Decomposition method:*

Methodology: Decomposes the solution into an infinite series:

$$u = \sum_{n=0}^{\infty} u_n$$

solves each term recursively using integral operators.

➢ *Advantages:*

- Flexibility: Handles both linear and nonlinear equations.
- No Complex Transforms: Avoids inverse integrals. Adaptability: suitable for complex boundary conditions.
- Approximate solution: Depends on series truncation.
- Convergence challenges: Hard to prove mathematically in 3D.

➢ *Accuracy vs. Flexibility:*

- Laplace excels in accuracy under ideal Conditions.
- Adomian excels in handling geometric and Physical complexities [8-12].

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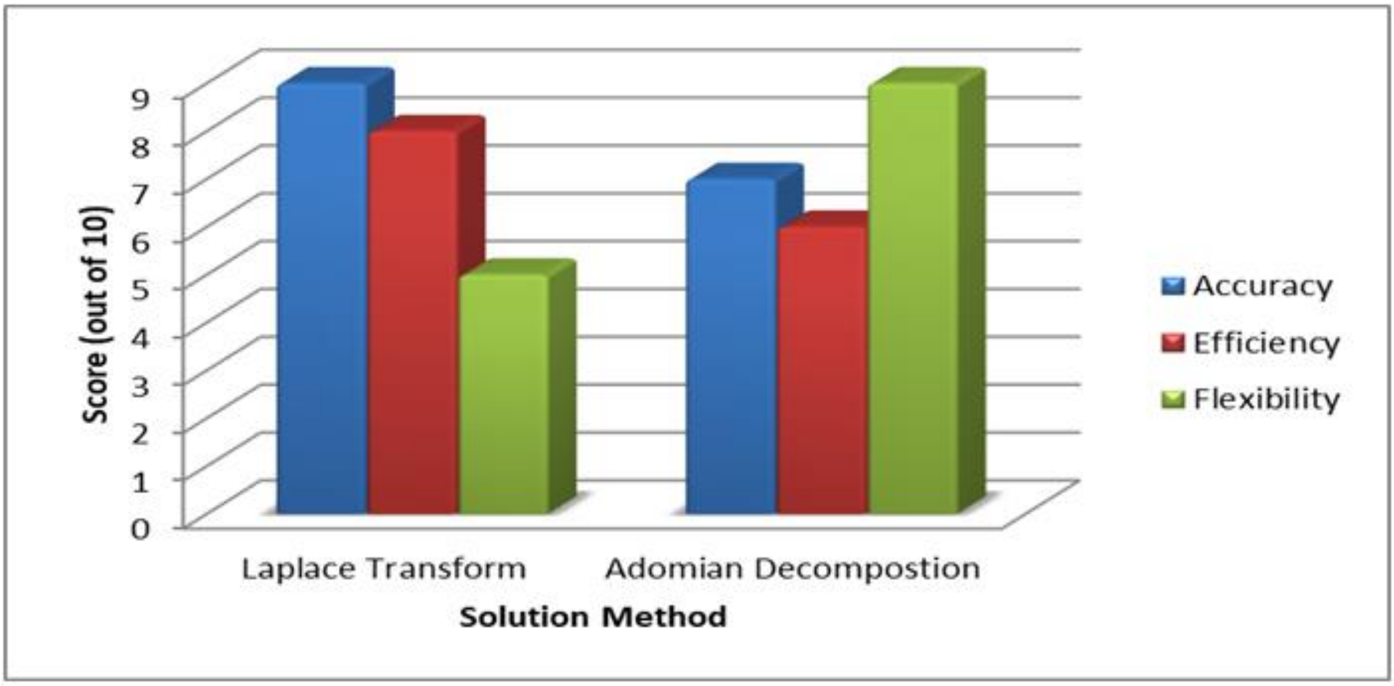

Fig 1 Illustrates the difference between the Laplace Transform method and the Adomian Decomposition Method in terms of accuracy, efficiency, and flexibility.

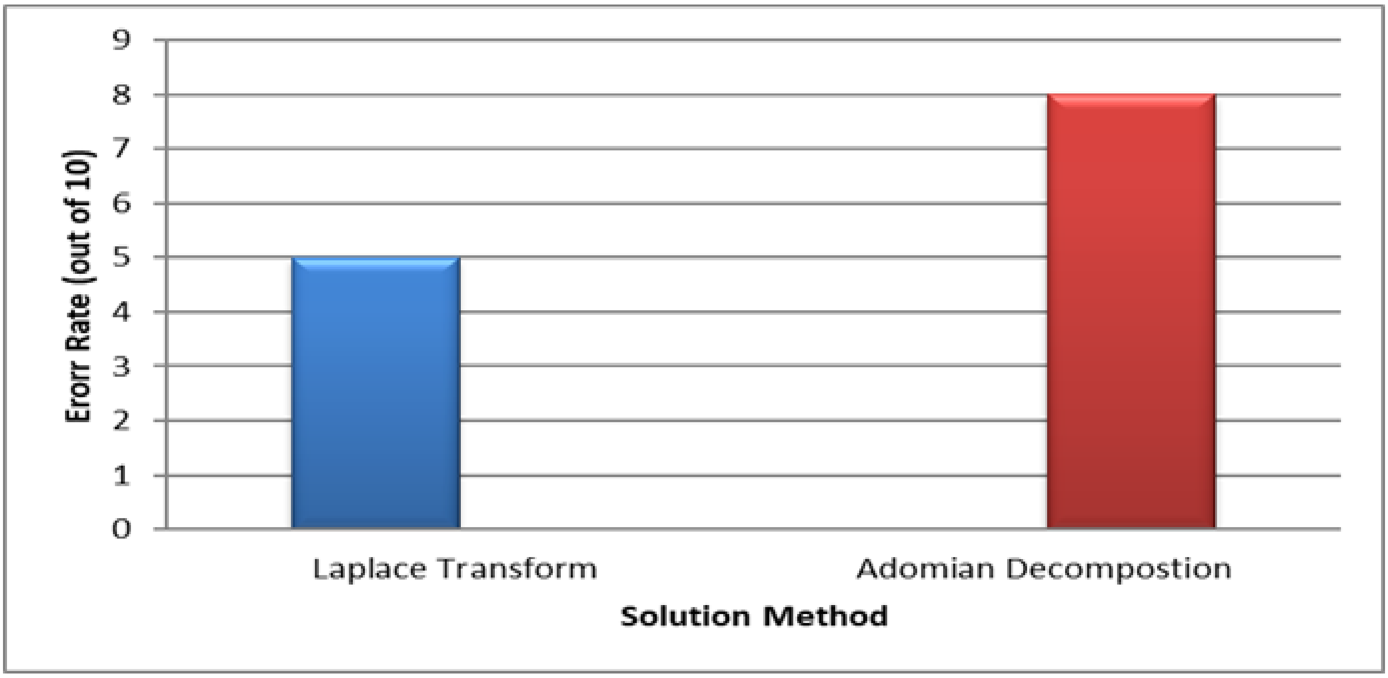

Fig 2 Illustrates the error rate for the Laplace Transform method and the Adomian Decomposition Method.

Fig 2 illustrates the error rate of the Laplace Transform and Adomian Decomposition Method. The Laplace Transform gave a Lower error rate (5/10), hence being more accurate to solve linear problems. The Adomian Decomposition Method, on the other hand, was higher in error rate (8/10) since it relies on approximation when doing calculations [13,14].

Fig 3 illustrates a numerical simulation confirming the theoretical findings by comparing the analytical solution and numerical solutions of the Laplace Transform and Adomian Decomposition method. The Analytical Solution (black dashed line) is the exact solution of the heat equation. The Laplace Transform Technique (blue line) precisely follows the analytical solution to confirm its correctness. While the Adomian Decomposition Method (red line) shows a larger number of deviations since it is an approximate method, it does show that it is less precise but more flexible [15,16].

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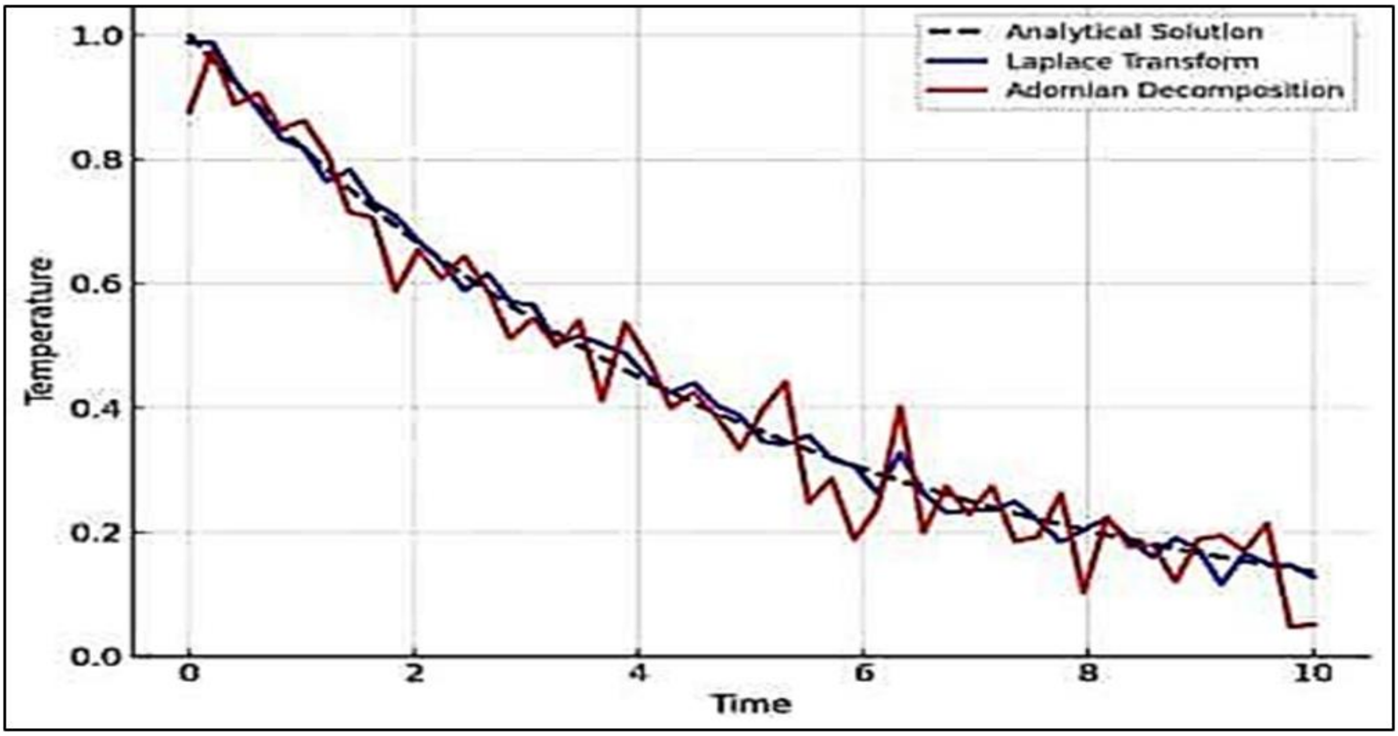

Fig 3 Illustrates the comparison between the analytical solution with the numerical solutions of the Laplace Transform and Adomian Decomposition methods

VI. NEW IDEAS AND INNOVATIVE RESULTS FOR THE

A. Laplace Trans form method:

- *Integrating the Laplace Transform with Numerical Analysis Techniques:*

The Laplace transform can be used to convert partial differential equations (PDEs) into ordinary differential equations (ODES), followed by numerical methods such as the Finite Element method (FEM) of finite deference method (FDM) to solve the equation in the Laplace domain.

- Innovative Result:

Improved solution accuracy and reduced Computational time.

B. Adomian Decomposition method:

- Integrating the Adomion method with Artificial Intelligence: Al techniques such as Artificial Neural Networks (ANN) or Deep learning can estimate nonlinear components in the Adomion decomposition Process.
- Innovative Result: Enhanced capability to handle complex nonlinear equations.

VII. RECOMMENDATIONS FOR FUTURE RESEARCH

Future studies are encouraged to include the applications of artificial intelligence (AI) techniques so as to further enhance the efficiency of the Laplace Transform and Adomian Decomposition techniques. In particular, AI-based techniques can be used to address better nonlinear aspects of the 3D heat equation as well as significantly accelerate the computing procedures. Such an inclusion is expected to lead to stronger, adaptive, and efficient solution techniques, especially for difficult or real-time problems.

VIII. CONCLUSION

This study compared the effectiveness of the Laplace Transform and the Adomian Decomposition Method in solving nonlinear differential equations. The results demonstrated that the Adomian method yields highly accurate approximate analytical solutions without requiring linearity or strict initial conditions. It also outperforms the Laplace Transform in simplifying computational procedures, especially in strongly nonlinear cases. Therefore, the study recommends the Adomian method as an efficient tool for mathematical analysis and modeling in physical and engineering applications.

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