# A Brief on Multi-Pantograph Ordinary Differential Equations with Constant Deviating Arguments: The Role of Lévy Noise in Almost Sure Exponential Stochastic Self-Stabilization

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**Abstract: This study investigates the role of Lévy noise in the almost sure exponential stochastic self-stabilization of multi-pantograph ordinary delay differential equations (MPODDEs). The equation is nonlinear and contains multi-pantograph terms as well as several constant time lags and as such, are typically unstable. The method applied involves the use of Lyapunov sample exponent function and a specialized convergence rate function technique suggested by Mao, (1997). It is demonstrated that if the noise driving force parameter of the stochastically perturbed equation is finite, then the new stochastic multi-pantograph ordinary delay differential equation (SMPODDE) is self - stabilized in an almost sure exponential sense. This phenomenon does not occur in the deterministic multi-pantograph ordinary delay differential equation where noise is absent.**

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## I. INTRODUCTION

Differential equations are used to sustain a relationship between functions and their derivatives. Where physical processes do not depend on the current state only, they are easily represented by means of delay differential equations (DDEs). The significance of these equations lies in their ability to describe processes with aftereffects (Unser (2020), Alferov (2018)).

A delay differential equation is a differential equation with deviating argument, this means, it is an equation that contains some functions and some of its derivatives are deviating argument values (Kolmanovskii & Myshkis, 1992). Situations in life are not usually free of disturbances which may be intrinsic or external. Most dynamic systems in real life are modeled by deterministic differential equations whenever they are not subject to randomness, but rather to changes occurring in their current state or parameters due to deterministic forces. Whenever such changes exist, the systems are said to be chaotic. Where real life situations are subject to internal or external disturbances, they are better modeled mathematically by means of stochastic delay differential equations (SDDEs).

The stability of dynamical systems becomes a research focus when these systems are subjected to performance criteria with the goal of accomplishing certain objectives. (Oguztoreli, 1979). Lyapunov introduced the concept of stability into the study of dynamic system in 1892 and explained the concept as insensitivity of a system to little changes in its initial state or parameters.

Different types of stability exist which form a better understanding of the concept viz: mean square stability, stochastic stability, moment stability, almost sure exponential stability, stability in probability, asymptotic stability etc. Stamova & Stamov, (2013) used Razumikhin methods and Lyapunov functions to establish the stability of the zero solution of differential equations with maximum. Xiao & Zhu, (2021) investigated the stability of unstable subsystems in switched stochastic delay differential systems. Li *et al*., (2021) examined the global Brownian motion-driven stabilization of multi-weight stochastic complex networks with time delay. Zhu & Huang, (2021) proved the stability of the general Brownian motion-driven class of stochastic delay nonlinear systems. Ngoc, (2021) created the new standards for the neutral stochastic functional differential equations' mean square exponential stability. Shen *et al*., (2021), using a general Levy process with non-Lipschitz coefficients, established sufficient condition for the mean square exponential instability of stochastic differential equations. They then designed a discrete-time feedback control in the drift part and achieved both mean square exponential stability

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and quasi-certain exponential stability for the controlled systems. Mao & Mao, (2017) examined if solutions to neutral stochastic functional differential equations with Lévy noise or jumps exist and are unique. Chen *et al.*, (2016) examined the neutral stochastic delay differential equations' exponential stability with time-varying delay. Liu, (2017) developed a semi group scheme for the drift part of the systems under consideration and with path-wise stability using a perturbation approach instead of moment stability, thereby establishing a theory regarding the property of almost sure path-wise exponential stability for a class of stochastic neutral functional simultaneous equations. Wei, (2019) used the Borel-Cantelli lemma, Burkholder-Davis-Gundy inequality, Holder inequality, Gronwall inequality, and the generalized Itô formula for Lévy stochastic integral to study the almost sure exponential stabilization of linear and nonlinear stochastic systems by stochastic feedback control with Lévy noise from discrete time systems.

From the above reviewed literatures, it can be seen that studies on the role of Lévy noise in the almost sure exponential stochastic stabilization of dynamical systems is scantly researched. The uniqueness of Lévy noise as contrast to Itô-noise lies in its ability to model fluctuations in a system by the presence of the Large Jumps and Compensated small Jumps, as will be described in this work. Motivated by some of the works of the authors mentioned above, we focus our attention on the study of stability behavior of deterministic multi-pantograph ordinary delay differential equations:

$$x^{|}(t) = f\Big(t, x(t), x(t-r_i)x(q(t),)x\big(q_2(t)\big), x\big(q_3(t)\big), \dots, x\big(q_m(t)\big)\Big),$$
$$t \geq 0, x(0) = \lambda \qquad (1.1)$$

Where $0 < q_i < 1, i = 1, 2, \dots, m$ are the pantograph functions, $r_i, i = 1, 2, \dots, m$ are constant deviating arguments and $f$ is a continuous function so that Eq. (1.1) is generally unstable (Mao (1997)). Eq. (1.1) is a functional differential equation that was first used for current collection in an electric locomotive system. (Ahmad and Mukhtar (2015)).

Pantograph differential equations are variant delay differential equations which have been widely studied since the 1850s. According to Rubab and Ahmad (2016), the pantograph is a device located on the electric locomotive. The first electric locomotive was made in 1851 and became commissioned in 1895. Taylor and Ockendon (1971) studied how electric current is collected by the pantograph of an electric locomotive and then developed a mathematical model of pantograph. Functional differential equations with proportional delays are usually referred to as pantograph equations.

The importance of pantograph differential equations emanates from the key role they play in understanding the asymptotic behavior and stability properties of complex systems such as electric locomotive, electrodynamics, astrophysics, nonlinear dynamic systems, electronic systems, population dynamics, probability theory on algebraic structures and quantum mechanics (Florescu (2014), Fristedt and Gray (2013) and Sezer et al. (2008). They are also applicable in computer science for image processing, signal processing, Simulations, sensor data, Neural network, see Unser (2020), Jordan and Turkington (2001).

- *The Present Article Studies the Deterministic Multi-Pantograph Ordinary Delay Differential Equation of the form:*

$$\left.\begin{aligned} x^{|}(t) &= \lambda x(t) + \sum_{i=1}^{k} \mu_i f\big(t, x(t), x(t-r_i), x(q_i t)\big), t \in [0, T] \\ x(t) &= \varphi(t), t \in [-\Gamma, 0] \end{aligned}\right\} \qquad (1.2)$$

Where, $f(.) = x^{\alpha}$, for $x > 0$ is an analytic function, $r_i > 0$ are fixed delays, $\lambda, \mu_i \in \mathbb{R}^+$, $q_i < q_{i-1} < q_{i-2} < 1, i = 1, 2, \dots, m \;\; t > 0$ *are the pantograph terms* , $\varphi \in C([-\Gamma, 0], \mathbb{R}^d)$ is the initial function and $\Gamma = \max_{0 \leq i \leq 1}\{r_i\}$. The equation is nonlinear and contains a multi-pantograph term as well as several constant time lags and as such, the solution is generally unstable Mao (1997), Hahn (1967).

By solution of Eq. (1.2), we refer to a continuous function $x \in C([\bar{t} - \Gamma, \infty), \mathbb{R})$ for some $\bar{t}$ which satisfies Eq. (1.2) together with its initial function for $t \geq \bar{t}$. Assume that for every initial datum $x(t_0) = \varphi(t_0) \in \mathbb{R}^d$, there exists a global solution $x(t, t_0, x_0)$. Assume also that $f\big(t, x(t_0 - r_i), x(q_i t)\big) = 0, \forall\, t \geq t_0$ so that the solution $x(t, t_0, x_0) \equiv 0$ corresponding to the initial datum $x(t_0) = \varphi(t_0)$. This solution is called the trivial solution or equilibrium position of Eq. (1.2).

The trivial solution $x(t, t_0, x_0)$ of Eq.(1.2) is stable if for every $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ *such that* $\big|x(t, t_0, x_0)\big| < \varepsilon, \forall\, t \geq t_0$ *and* $|x_0| < \delta$.

A multi-pantograph delays differential equation with several deviating argument is a type of pantograph differential equation that incorporates proportional delays and multiple pantograph terms. They are particularly useful for modeling complex systems with multiple dependencies at different time scales like Population dynamics with time dependent reproduction rates, control systems with multiple feedback delays, and also mechanical systems with proportional interactions (Valdimirov *et al.*, 2021; Kovalev

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and Viktorov, 2020; Klinshov and Zlobin, 2023). It is an extension of the standard pantograph equation as in Eq.(1.2) above.

A stochastic differential equation (SDEs) is an equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process. That is SDEs contain a variable which represents random white noise calculated as the derivative of Brownian motion. Uncertain stock prices in finance are just one example of the many phenomena that have been modelled using stochastic differential equations (Merton, 1976). Atonuje *et al.*, (2024) examined how a multiplicative Ito-type Brownian noise can stochastically stabilize the evolution of a Volterra functional–describing optimum control dynamical system, which is characterized by an unstable nonlinear classical delay differential equation. In order to create a stochastic optimum control system, the authors introduced a multiplicative Brownian noise into the equation. The system became almost surely exponentially self-stabilized under specific conditions and with a tiny enough time delay when the noise scaling parameter in the stochastic optimum control delay differential equation is replaced with a finite integral expression.

However, there are other types of random behavior that are possible, such as jump processes which are seen in the Lévy noise.

The Lévy process is a type of stochastic process widely used in probability theory and mathematical finance (Atonuje, 2017). It is named after the French mathematician Paul Lévy. In essence, it is a continuous-time random process that exhibits independent and stationary increments.

➢ *A Lévy Process $X = X(t): t \geq 0$ Stochastic Process Such that:*

- $X_0 = 0$ almost surely, with positive probability. Typically, the Lévy process begins at zero.
- $X$ has stationary increment i.e $(X▯-X▯\ for\ s < t \equiv (X_{(t-s)})$. The distribution of the process's increments depends only on the length of the time interval, not on the starting point
- $X_{t_2} - X_{t_1},\ X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independently mutually increasing for each $0 \leq t_1 < t_2 < \cdots < t_n < \infty$.
- $\{X(t)\}_{t\geq 0}$ is continuous in probability, i.e., given any $\varepsilon > 0$,

$$P(|X(t+s) - X(t)| > \varepsilon) = 0$$

➢ *Eq. (1.2) is now Perturbed by Lévy Noise into a Stochastic Multi-Pantograph Ordinary Delay Differential Equation (SMPODDE) of the form:*

$$\begin{cases} dx(t) = \left(\lambda x(t) + \sum_{i=1}^{n} \mu_i f\big(t, x(t), x(t-r_i), x(q_i t)\big)\right) dt + \\ \sigma\left(\left[\sum_{k=1}^{m} G_k x(t) dB_k(t)\right] + \int_{|y|<\Delta} D(y)\, x(t) \bar{N}(dt, dy) + \int_{|y|\geq\Delta} E(y)\, x(t) N(dt, dy)\right) \\ x(t) = \varphi(t), t \in [-\Gamma, 0] \end{cases} \quad (1.3)$$

For all $t \geq t_0$, where $G_k \in M_d(\mathbb{R})\ for\ 1 \leq k \leq m$, D and E are suitable functions. $D: \mathbb{R}^m \to M_d(\mathbb{R})$, $E: \mathbb{R}^m \to M_d(\mathbb{R})$. $\sigma$ is the noise driving force parameter which measures the intensity of the fast-fluctuating Lévy noise. The positive number $\Delta$ plays the role of separating small jumps (which are compensated) from large jumps (which are not compensated). N(.) is the Poisson process with compensation $\bar{N}$ and $B_k(t) = (B_1(t),\ B_2(t), \ldots, B_m(t))^T, t \geq 0$ is an m-dimensional Brownian motion defined on a complete probability triple $(\Omega, F, P)$, with filtration $\{F_t\}_{t\geq 0}$ which is right continuous and each $F_t$ contains all P-null sets in F with expectation $E(x) = \int_\Omega x dP$ , where $E$ denotes expectation.

- *Eq (1.3) Can be Written in the Integral form as*

$$x_n(t) = \lambda x(0) \int_{t_0}^{t} \mu f_n\big(s, x_n(s)\big), x_n(s - r_i), x(q_i(s)\, ds +$$

$$\sigma\left[\int_{t_0}^{t} G_k x_n(s) dBs + \int_{t_0}^{t} \int_{|y|<\Delta} D_n\big(y, x_n(s)\big) \bar{N}(ds, dy) + \int_{t_0}^{t} \int_{|y|\geq\Delta} E_n\big(y, x_n(s)\big) N(ds, dy)\right]$$

Recall that every Levy process is the sum of a Brownian motion with drift and another independent random variable. In the above case the other random process is Poisson. This stochastic sum is called the Lévy-Ito decomposition and is referred to as a stochastic sum of independent Poisson random variable.

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➢ *In the Present Article, we Focus our Attention on Providing an Answer to the Following Questions:*

- if under certain conditions, a deterministic multi-pantograph delay differential equation which is generally unstable, is stochastically perturbed by Levy noise, can the presence of a sufficiently strong noise driving force parameter, stochastically stabilize the system in an almost sure *(a.s.)* exponential sense?

- Under these same conditions, can the comparable deterministic system where noise is absent be stabilized or would it remain unstable?

This research is thus focused on proposing to establish the contribution of Levy noise to the almost sure exponential stochastic self-stabilization in a specific type of deterministic pantograph delay differential equation described in (1.2) above. The approach used was first suggested in Mao (1997) and it utilizes the concept of Lyapunov sample exponent function.

## II. METHODOLOGY AND PRELIMINARIES

➢ *The Following Assumptions are Imposed on the Functions* $f, D, E$ *and* $\varphi$:

- $f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}, D: \mathbb{R}\times\mathbb{R} \to \mathbb{R}, E: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ *and* $\varphi: [-\Gamma, 0] \to \mathbb{R}$ are such that f, D and E are continuous functions and fulfill the uniform Lipchitz condition. More specifically, there exist some positive constants $K_1, K_2, K_3, K_4$ $K_5$and $K_6$ such that $\forall\ a_1, a_2, b_1, b_2, \in \mathbb{R}$

$$|f(a_1, b_1) - f(a_2, b_2)| \le K_1\,|a_1 - a_2| + K_2\,|b_1 - b_2|$$

$$|D(a_1, b_1) - D(a_2, b_2)| \le K_3\,|a_1 - a_2| + K_4\,|b_1 - b_2|$$

$$|E(a_1, b_1) - E(a_2, b_2)| \le K_5\,|a_1 - a_2| + K_6\,|b_1 - b_2|$$

- $f, D$ *and* $E$ fulfill the Linear growth bound condition. More specifically, there exist positive constants $L_1, L_2$ *and* $L_3$ such that for all $a, a_1, b, b_1, c, c_1 \in \mathbb{R}$ such that

$$|f(a, a_1)|^2 \le L_1(1 + |a|^2 + |a_1|^2)$$

$$|D(b, b_1)|^2 \le L_2(1 + |b|^2 + |b_1|^2)$$

$$|E(c, c_1)|^2 \le L_3(1 + |c|^2 + |c_1|^2)$$

- The initial function $\varphi$ is Holder-continuous with exponent $\beta$. More specifically, there exists a positive constant $K_6$ such that $\forall\ t, s \in [-\Gamma, 0]$ , $\varepsilon(|\varphi(t) - \varphi(s)|^p) \le K_6\,|t - s|^{p\beta}, p = 1, 2.$

The conditions $(i)$ and $(ii)$ ensures the existence of solutions of Eq. (1.3). For detailed understanding of the concept of expectation, existence and uniqueness of solutions of differential equations, the reader is referred to Arnold (1974).

By solution of the SMPODDE (1.3), one refers to an $\mathbb{R} - value\ x(t): [-\Gamma, 0] \times \Omega \to \mathbb{R}$, called a strong solution of Eq. (3) if it is a measurable sample – continuous process such that $x|[0, T]$ is $\{F_t\}_{0\le t\le T} - adapted,\ f, D\ and\ E$ are continuous functions and $x(t)$ satisfies Eq. (3) almost surely as well as the initial condition $x(t) = \varphi(t), t\epsilon\,[-\Gamma, 0]$. $x(t)$ is said to be unique if there exists any other solution $\bar{x}\,(t)$ is indistinguishable from $x(t)$. that is, $P(x(t) = \bar{x}\,(t), for\ all\ t \in [-\Gamma, 0]\,) = 1.$

Eq. (1.3) can be stabilized in an almost sure *(a.s.)* exponential sense by Lévy noise provided that the noise driving force parameter $\sigma$ is sufficiently large, i.e., $\sigma < \infty$. In this section, a carefully chosen integral expression is used to replace $\sigma$. that is, $\sigma = \int_0^t |\eta(s)x(s)|^p ds,\ p \in \mathbb{R}_+$ to obtain Eq. (1.4) below, where $\eta(s)$ is a continuous $\mathbb{R}^{m\times d} - valued$ function with the property $\|\eta(s)\| \le Qe^{\delta t}, \forall t \ge 0$. $\eta(s)$ acts to restrain the solution $x(t)$ so that it must not vanish into infinity at t. $\eta(t)$ in this capacity is called a convergence rate function.

$$\begin{cases} dX(t) = \left(\lambda X(t) + \sum_{i=1}^{n} \mu_i f\big(t, X(t), X(t - r_i), X(q_i t)\big)\right)dt + \\ \left(\int_0^t |\eta(s)x(s)|^p ds\right)\left(\left[\sum_{k=1}^{m} G_k X(t) dB_k(t)\right] + \int_{|y|<\Delta} D(y)\,x(t)\bar{N}(dt, dy) + \int_{|y|\ge\Delta} E(y)\,x(t)N(dt, dy)\right) \\ X(t) = \varphi(t), t \in [-\Gamma, 0] \end{cases} \quad (1.4)$$

That is, the noise driving force parameter $\sigma$ in Eq. (1.3) is carefully replaced by $\int_0^t |\eta(s)x(t, w)|^p ds$, where $\eta(s)$ is the convergence rate function.

For Eq. (1.4) to be almost surely exponentially self-stabilized, we impose the condition that for some $w \in \Omega, \sigma = \int_0^t |\eta(s)x(t, w)|^p ds < \infty$. This means that the noise driving force parameter must be finite. The main focus of this work is to carefully choose a noise driving force parameter $\sigma < \infty$ and derive results which ensure an $a.s.$ exponential stability of Eq. (1.3).

The noise driving force parameter ensures that the noise intensity is sufficiently large enough at which point the SMPODDE is said to be stabilized (Mao,1997). This is only possible due to the presence of Lévy noise. The comparable deterministic Eq. (1.2), where Lévy noise is absent remains unstable in general. Almost sure exponential Stability is here induced by the presence of Lévy noise.

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➢ *Definition 1(The Trivial Solution of SMPODDE)*

Assume that $\{X(t)\}_{t\geq 0}$ is the solution of equation (1.3). Suppose that $f(0,0,0,t) \equiv 0, g(0,0,0) \equiv 0 \; \forall \; t > 0$. It follows that equation (1.3) has the solution $X(t) \equiv 0$ corresponding to the initial datum $X(t_0) \equiv 0$ is called the trivial or zero solution of equation (1.3).

➢ *Definition 2(Almost Sure Exponential Stability of SMPODDE)*

The trivial solution $x(t;\, t_{0,}x_0)$ of equation (1.3) is said to *almost surely (a. s.)* exponentially stable if $\underset{t\to\infty}{LimitSup}\frac{1}{t}log|x(t;\, t_0,x_0)| < 0, a.s. for\ all\ x_0 \in \mathbb{R}^d$.

where, $\underset{t\to\infty}{LimitSup}\frac{1}{t}log|x(t;\, t_0,x_0)|$ is called 'the Lyapunov sample exponent function.

The following Assumptions, Lemmas and Theorems, are based on the Ito formula called exponential martingale inequality. It is useful to the proof of the main result.

## III. THE MAIN RESULTS

➢ *From Conditions i and ii Above, Which Show that Functions are Locally Lipschitz Continuous, we Impose the Following Conditions:*

- *Assumption*

✓ *Hypothesis (H1):*

Let $M$ be a symmetric –definite $d \times d - matrix$. Assume the there exist some positive constants $K, \theta, \omega\ such\ that\ 2\omega > \theta,$ then

$$\left|x^T M f\left(\left(t, x(t), x(t-r_i), x(q_i t)\right)\right)\right| \leq K|x|^2$$

$$trace\left(G_k^T(x,t) M G_k(x,t)\right) trace\left(D^T(x,t) M D(x,t)\right) trace\left(E^T(x,t) M E(x,t)\right) \leq \theta x^T\, Mx$$

$$\left|\left(x^T M G_k(x,t)\right)\left(x^T M D(x,t)\right)\left(x^T E(x,t)\right)\right|^2 \geq \omega |x^T\, Mx|^2,$$

$$\forall\, t \geq 0\ and\ x \in \mathbb{R}^d.$$

This hypothesis guarantees that for all sufficiently large $\sigma$, the trivial solution of Eq. (1.3) ia almost surely exponentially stable. Eq. (1.3) is regarded as the stochastically stabilized system of the deterministic multi-pantograph delay differential equation in (1.2), which is generally unstable. In other words, Eq. (1.2) is stabilized by Lévy noise provided that the noise intensity is Large enough. The trivial solution of Eq. (1.3) stabilizes itself in an almost sure exponential sense, if $\sigma$ is replaced by a carefully chosen finite expression.

- *Lemma*

Suppose that assumption (H1) hold. Then the solution of equation (1.4) satisfies the property that

$$p\{x(t, x0) \neq 0 \, \forall\, t \geq 0\} = 1$$

provided that $x_0 \neq 0$.

- *Proof:*

Suppose the assertion is not true. Then ∃ some $x_0 \neq 0 \ni P(\mu < \infty) > 0$, where $\mu$ represents the time of first reaching state zero, i.e

$$\mu = \inf\{t \geq 0 : x(t) = 0\}.$$

We now find some $\bar{t} > 0$ and $\phi > 0$ large enough to ensure that $P(D) > 0$, where

$$D = \{\partial : \mu \leq \bar{t}\ and\ |x(t)| \leq \phi - 1\, \forall\, 0 \leq t \leq \mu\}$$

For each 0< $\beta < |x_0|$, define

$$\mu_\beta = \inf\{t \geq 0 : x(t) \leq \beta\ or\ |x(t)| \geq \phi\}.$$

Therefore by Itôs Formula, for $0 \leq t \leq \bar{t}$,

$$E\left[\left|x^T(t \wedge \mu_\beta) M x(t \wedge \mu_\beta)\right|^{-1}\right] \leq |x_0^T M x_0|^{-1} + 2E \int_0^{t\wedge\mu_\beta} |x^T(s) M x(s)|^{-2} |x^T(s) M f(x(s), s| ds$$

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$$+4E\int_0^{t\wedge\mu_\beta} |x^T(s)Mx(s)|^{-3}|x^T(s)M[(G_k(x(s),s),D(x(s),s),E(x(s),s)]|^2\left(\int_0^s |r(u)x(u)|^p\,du\right)^2 ds$$

By (H1) it becomes evident that

$$E\left[\left|x^T(t\wedge\mu_\beta)Mx(t\wedge\mu_\beta)\right|^{-1}\right] \le |x_0^T Mx_0|^{-1} + \alpha E\int_0^{t\wedge\mu_\beta} |x^T(s)Mx(s)|^{-1}ds$$

$$\le |x_0^T Mx_0|^{-1} + \alpha\int_0^{t\wedge\mu_\beta} E\left[\left|x^T(s\wedge\mu_\beta)Mx(s\wedge\mu_\beta)\right|^{-1}\right]ds$$

Where $\alpha$ represents a constant dependent on $K, w, \theta, \bar{t}, \mu, \phi, M$ $but\ independent$ of $\beta$.

Applying gronwalls inequality we have,

$$E\left[\left|x^T(\bar{t}\wedge\mu_\beta)Mx(\bar{t}\wedge\mu_\beta)\right|^{-1}\right] \le |x_0^T Mx_0|^{-1}e^{\alpha\bar{t}}$$

Note that $\partial \in D$, and $\mu_\beta \in \bar{t}$ and $|x(\mu_\beta)| = \beta$, it then quickly follows that

$$(\beta^2\|M\|)^{-1}P(D) \le |x_0^T Mx_0|^{-1}e^{\alpha\bar{t}}$$

As $\beta \to 0$, we see that $P(D) = 0$, which contradicts the definition of D.

The proof is hence complete.

We can now establish the main results by making a clearer hypothesis which will form the condition on the convergence rate function $\eta(t)$.

- *Assumption*

✓ *Hypothesis (H2):*
There exists a pair of constants $Q > 0$ $and$ $\delta \ge 0$ such that

$$\|\eta(t)\| \le Qe^{\delta t} for\ all\ t \ge 0$$

We point out that the local Lipschitz continuity of the coefficients $f, D, E$ as speculated in conditions $i$ and $ii$ above as well as the standing hypothesis (H1) guarantee the existence and uniqueness of the solution $x(t;\ t_0, x_0)$ of Eq (1.4). (Detailed proof can be found in Mao, 1996).

It is also clear that Eq.(1.4) admits a trivial solution $x(t;0) = 0$ for hypothesis (H2) implies that $F(0,0,\ldots,0) = 0, D(0,0,\ldots,0) = 0\ and\ E(0,0,\ldots,0,0) = 0$.

To now prove the Main result, we establish our Theorem.

- *Theorem*
Let (H1) and (H2) hold. Then the solution of equation (1.4) has the property

$$\int_0^\infty |\eta(t)x(t,x_0)|^p dt < \infty, \qquad a.s.\ holds\ for\ all\ x_0 \in \mathbb{R}^d \qquad (1.5)$$

✓ *Proof*
Since hypothesis (H1) guarantees $x(t;0) \equiv 0$, one only needs to show that (1.5) holds for $x_0 \neq 0$

$$i.e\ \sigma = \int_0^\infty |\eta(t)x(t,x_0)|^p dt \quad < \infty,\ a.s.\ holds\ for\ all\ x_0\ \neq 0\ .$$

By the property in Lemma 3.1, i.e., $P(x(t,x_0)\ \neq 0,\ \forall\, t\ \geq 0\,) = 1$, provided $x_0\ \neq 0, a.s.$, the solution $x(t,x_0)\ \neq 0, \forall\, t\ \geq 0$ almost surely.

Now assume on the contrary that Eq (1.5) is not true, it must be that there exists a given $x_0\ \neq 0, a.s.$ for which $P(\Omega^*) > 0$, $for\ some\ \Omega^*\ \subseteq\ \Omega$, where

$$\Omega^* = \left\{ \mathrm{w} \in \Omega\colon \int_0^\infty |\eta(t)x(t,x_0)|^p dt\ = \infty,\ \ a.s. \right\}$$

Applying Ito's formula and Hypothesis (H1). It is easily verified that for some given $t \geq 0$

$$log\big(x^T(t)Mx(t)\big)\ \leq log(x_0^T\ Mx_0) + \frac{2Kt}{\sqrt{\lambda_{min}(M)}} + \theta \int_0^t \left( \int_0^s |\eta(v)x(v)|^p dv \right)^2 ds$$

$$-2\int_0^t \left(\int_0^s |\eta(v)x(v)|^p dv\right)^2 \left[ \frac{|x^T(s)MG_k(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} \frac{|x^T(s)MD(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} \frac{|x^T(s)ME(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} \right] ds + Q(t)\ (1.6$$

Where, $\sqrt{\lambda_{min(M)}}$ is the smallest Eigen value of the matrix M and Q(t) is the expression

$$Q(t) = 2\int_0^t \left( \int_0^s |\eta(v)x(v)|^p dv \right)^2 \frac{|x^T(s)MG_k(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} \frac{|x^T(s)MD(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} \frac{|x^T(s)ME(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} \left( \left[ \sum_{k=1}^m G_k x(t) dB_k(t) \right] + \int_{|y|<\Delta} D(y)\, x(t)\bar{N}(dt,dy) + \int_{|y|\geq\Delta} E(y)\, x(t)N(dt,dy) \right) ds$$

is a continuous Martingale vanishing at $t\ =\ 0$. Assume $k\ =\ 1, 2, \ldots$, it follows from the exponential Martingale inequality that

$$P\left(\mathrm{w}\colon \underset{0\ \leq t \leq k}{Sup} \left[ Q(t) - \frac{2\omega-\theta}{8\omega} \langle Q(t),\qquad Q(t)\rangle \right] > \frac{8\omega logk}{2\omega-\theta} \right)\ \leq \frac{1}{k^2}$$

Where

$$\langle Q(t),Q(t)\rangle = 4\int_0^t \left( \int_0^s |\eta(v)x(v)|^p dv \right)^2 \left[ \frac{|x^T(s)MG_k(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} \frac{|x^T(s)MD(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} \frac{|x^T(s)ME(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} \right] ds$$

Applying Borel- Cantelli Lemma, one gets that given almost all w ∈ Ω , there exists a random integer $k_1(\mathrm{w})\ such\ that\ \forall\ k\ > k_1(\mathrm{w})$,

$$\underset{0\ \leq t \leq k}{Sup} \left[ Q(t) - \frac{2\omega-\theta}{8\omega} \langle Q(t),Q(t)\rangle \right]\ \leq\ \frac{8\omega logk}{2\omega-\theta}$$

That is, for $0\ \leq t \leq k$,

$$Q(t)\ \leq\ \frac{8\omega logk}{2\omega-\theta} + \frac{2\omega-\theta}{8\omega} \langle Q(t),Q(t)\rangle$$

$$\leq\ \frac{8\omega logk}{2\omega-\theta} +\ \frac{2\omega-\theta}{8\omega} \int_0^t \left(\int_0^s |\eta(v)x(v)|^p dv\right)^2 \frac{|x^T(s)MG_k(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} \frac{|x^T(s)MD(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2}\ \frac{|x^T(s)ME(x(s),s),s|^2}{\left(x^T(s)Mx(s)\right)^2} ds \qquad (1.7)$$

Substituting (1.7) into (1.6) and applying Hypothesis (H1) one gets

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$$log\left(x^T(t)Mx(t)\right) \le log(x_0^T Mx_0) + \frac{2Kt}{\lambda_{min}(M)} + \frac{8\omega logk}{2\omega - \theta}$$

$$-\frac{2\omega-\theta}{2}\int_0^t\left(\int_0^s |\eta(v)x(v)|^p dv\right)^2 ds \qquad (1.8)$$

for all $0 \le t \le k, k \ge k_1$ almost surely. By the definition of $\Omega^*$, it is clear that for every $w \in \Omega^*$, there exists a random integer $k_2(w)$ such that:

$$\int_0^t |\eta(s)x(s)|^p ds \ge \frac{\sqrt{\frac{4k}{\lambda_{min(M)}} + 4\delta + 8}}{2\omega - \theta}, \quad \text{for all } t \ge k_2 \qquad (1.9)$$

It then follows from (1.8) and (1.9) and almost all $w \in \Omega^*$ that, if $K - 1 \le t \le k, k \ge k_1 \vee (k_2 + 1)$,

$$log\left(x^T(t)M(t)\right)$$

$$\le log(x_0^T Mx_0) + \frac{2Kk}{\lambda_{min(M)}} + \frac{8\omega logk}{2\omega-\theta} - \frac{2\omega-\theta}{2}\int_0^t\left(\int_0^s |\eta(v)x(v)|^p dv\right)^2 ds$$

$$\le log(x_0^T Mx_0) + \frac{2Kk}{\lambda_{min(M)}} + \frac{8\omega logk}{2\omega-\theta} - \left(\frac{2K}{\lambda_{min(M)}} + 2\delta + 4\right)(K - 1 - k_2)$$

$$= log(x_0^T Mx_0) + \frac{2K(k_2+1)}{\lambda_{min(M)}} + \frac{8\omega logk}{2\omega-\theta} - 2(\delta+2)(k - 1 - k_2)$$

and as such,

$$\frac{1}{t}log\left(x^T(t)Mx(t)\right) \le \frac{1}{k-1}log(x_0^T Mx_0) + \frac{2K(k_2+1)}{\lambda_{min(M)}} + \frac{8\omega logk}{2\omega-\theta} - 2(\delta+2)(k-1-k_2)$$

which then follows that

$$\lim_{t\to\infty} Sup \frac{1}{t}log\left(x^T(t)Mx(t)\right) \le -2(\delta+2) \; for\ almost\ all\ \ w \in \Omega^* \qquad (1.10)$$

Thus, for almost all sample path $w \in \Omega^*$ , there exist a random $k_3(w)$ as such,

$$\frac{1}{t}log\left(x^T(t)Mx(t)\right) \le -2(\delta+2) \qquad for\ all\ t \ge k_3$$

And hence,

$$|x(t)| \le \frac{e^{-(\delta+2)t}}{\sqrt{\lambda_{min(M)}}}, \forall\, t \ge k_3$$

. Now applying hypothesis (H2) for almost all $w \in \Omega^*, t \to \infty$

$$\int_0^\infty |\eta(t)x(t)|^p dt \le \int_0^{k_3} Q^p e^{p\delta t}|x(t)|^p\, dt + \int_{k_3}^\infty \frac{Q^p e^{-pt}}{|\lambda_{min}(M)|^{p/2}} dt < \infty$$

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This is a contradiction by the definition of $\Omega^*$ and hence, the property in equation (1.5) must hold and the proof is complete.

$$\int_0^\infty |\eta(t)x(t,x_0)|^p dt \quad < \infty,\ a.s. \text{ holds.}$$

- *Theorem:* (Precise Estimate for the Solution)

Let hypothesis (H1) and (H2) hold, then for every $x_0 \in \mathbb{R}^d$.

$$\int_0^\infty |\eta(t)x(t)|^p dt \le \sqrt{\frac{2K}{(2w-\theta)\lambda_{min}(M)}} \tag{1.11}$$

*or*

$$\lim_{t\to\infty} Sup \frac{1}{t} Log(|x(t)|) < 0 \tag{1.12}$$

Holds for almost all w ∈ Ω.

✓ *Proof.*

Fix $x_0 \neq 0$, and continue representing $x(t,x_0)\ as\ x(t)$, we define

$$\widetilde{\Omega} = \left\{ \mathrm{w} \in \Omega: \int_0^\infty |\eta(t)x(t)|^p dt > \sqrt{\frac{2K}{(2w-\theta)\lambda_{min}(M)}} \right\}$$

We need to show that (1.12) holds for all $\mathrm{w} \in \widetilde{\Omega}$. Suppose $j = 1,2, \ldots$, define

$$\widetilde{\Omega}_j = \left\{ \mathrm{w} \in \widetilde{\Omega}: \int_0^\infty |\eta(t)x(t)|^p dt > (1+j^{-1}) \sqrt{\frac{2K}{(2w-\theta)\lambda_{min}(M)}} \right\}$$

We assert that $\widetilde{\Omega} = \cup_{j=1}^{\infty} \widetilde{\Omega}_j$. and hence we just need to demonstrate that for each $j \ge 1$, Eq(1.12) holds for all $\mathrm{w} \in \widetilde{\Omega}$. We fix any $j \ge 1$. And in the same way as (1.8) we can deduce that for each $\mathrm{w} \in \Omega - \widehat{\Omega}$, where $\widehat{\Omega}$ is a P-null set, there exist an integer $k_4(w)$ such that

$$log\big(x^T(t)Mx(t)\big) \le log(x_0^T M x_0) + \frac{2Kt}{\lambda_{min}(M)} + \frac{4\omega(1+j^{-1})logk}{2\omega-\theta}$$

$$-\frac{2\omega-\theta}{1+j^{-1}} \int_0^t \left( \int_0^s |\eta(v)x(v)|^p dv \right)^2 ds \tag{1.13}$$

For all $0 \le t \le k, k \ge k_4$. Conversely, for every $\mathrm{w} \in \widetilde{\Omega}$, there exists an integer $k_5$ such that

$$\int_0^\infty |\eta(s)x(s)|^p ds \le (1+j^{-1}) \sqrt{\frac{2K}{(2w-\theta)\lambda_{min}(M)}} \quad for\ all\ t \ge k_5 \tag{1.14}$$

It then follows from (1.13) and (1.14) that for all $\mathrm{w} \in \Omega - \widehat{\Omega}$, if $k-1 \le t \le k, k_4 \vee (k_5+1)$,

$$log\big(x^T(t)Mx(t)\big) \le log(x_0^T M x_0) + \frac{2K(k_5+1)}{\lambda_{min}(M)} + \frac{4\omega(1+j^{-1})logk}{2\omega-\theta} - \frac{2K}{j\lambda_{min}(M)}(k-1-k_5)$$

Which implies that

$$\lim_{t\to\infty} Sup \frac{1}{t} log\big(x^T(t)Mx(t)\big) \le -\frac{2K}{j\lambda_{min}(M)}\ \forall\ \mathrm{w} \in \widetilde{\Omega} - \widehat{\Omega}.$$

And consequently

$$\lim_{t\to\infty} Sup \frac{1}{t}\log(|x(t)|) \leq -\frac{K}{j\lambda_{min}(M)} < 0 \;\; \forall \;\; w \in \tilde{\Omega} - \hat{\Omega}.$$

This Justifies the condition used to verify the property (1.5) in Theorem 5.1, that if

$$\int_0^\infty |\eta(t)x(t)|^p dt > \sqrt{\frac{2K}{(2w-\theta)\lambda_{min}(M)}}$$

Then the solution $x(t; x_0)$ tends to zero exponentially, and under Hypothesis (H1) and (H2) 1.4 is almost surely exponentially stable provided that the noise driving force parameter satisfies

$$\sigma > \sqrt{\frac{2K}{(2w-\theta)\lambda_{min}(M)}}.$$

## IV. CONCLUSION

In this study, we established the almost sure exponential stability of the nonlinear stochastic multi-pantograph ordinary delay differential equations (MPO$DDE$s) with constant delay or time lag. Our findings reveal that, it is possible to stabilize an unstable dynamical systems in the presence of sufficiently Large Lévy noise, measured with the noise driving force parameter $\sigma$. The noise driving force parameter was carefully chosen and hypothesis established to satisfy the finite convergence rate condition $\int_0^\infty |\eta(t)x(t,x_0)|^p dt \;\; < \infty,\ a.s.$, which form the Lyapunov sample exponent. The sampled Lyapunov exponent must always be finite for the resulting stochastic system to be stabilized by Lévy noise. If the noise term is sufficiently large enough, the diffusion function is absent, the deterministic system will continue to be unstable. In particular, the new theorem enables us to stabilize the unstable deterministic system in an almost sure exponential sense. The method of Lyapunov sample exponents together with stochastic perturbation were used to achieve stability result. The crucial condition which ensures the $a.s.$ exponential stability of the trivial solution of the resulting (SMPODDE) is $\int_0^\infty |\eta(t)x(t,x_0)|^p dt \;\; < \infty,\ a.s.$ In the comparable deterministic pantograph differential (MPODDE) where Levy noise is absent, one sees that for some w $\in \Omega, \int_0^\infty |x(t,w)|dt = \infty$ and at that point, the trivial solution of the deterministic can never be stable.

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